

A nonstandard uniform functional limit law for the increments of the multivariate empirical distribution function

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Abstract

Let $(Z_i)_{i \geq 1}$ be an independent, identically distributed sequence of random variables on \mathbb{R}^d . Under mild conditions on the density of Z_1 , we provide a nonstandard uniform functional limit law for the following processes on $[0, 1]^d$:

$$\Delta_n(z, h_n, \cdot) := s \mapsto \frac{\sum_{i=1}^n 1_{[0, s_1] \times \dots \times [0, s_d]} \left(\frac{Z_i - z}{h_n^{1/d}} \right)}{c \log n}, \quad s \in [0, 1]^d,$$

along a sequence $(h_n)_{n \geq 1}$ fulfilling $h_n \downarrow 0$, $nh_n \uparrow$, $nh_n / \log c \rightarrow c > 0$. Here z ranges through a compact set of \mathbb{R}^d . This result is an extension of a theorem of Deheuvels and Mason [5] to the multivariate, non uniform case.

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1 Introduction and statement of the result

In this paper, we consider an independent, identically distributed sequence of random vectors $(Z_i)_{i \geq 1}$ having a density f on an open set $O \subset \mathbb{R}^d$. We make the following assumption on f :

(Hf) f is continuous and strictly positive on O .

Throughout this article, $s, s' \in \mathbb{R}^d$, we shall write $s \prec s'$ when $s_i \leq s'_i$ for each $i = 1, \dots, d$. Intervals and semi intervals are implicitly understood as product of intervals or semi intervals, namely

$$\begin{aligned} [s, s'] &:= \{u \in \mathbb{R}^d, s \prec u \prec s'\} \\ &= [s_1, s'_1] \times \dots \times [s_d, s'_d], \quad s = (s_1, \dots, s_d), \quad s' = (s'_1, \dots, s'_d). \end{aligned} \quad (1.1)$$

We shall also write $a \prec s$ (resp. $s \prec a$) for $s \in \mathbb{R}^d$ and $a \in \mathbb{R}$ when $a \leq s_i$ (resp. $s_i \leq a$) for each $i = 1, \dots, d$. For fixed $0 < h < 1$ and $z \in O$, we define the following process on $[0, 1]^d$:

$$\Delta_n(z, h, s) := \frac{1}{n} \sum_{i=1}^n 1_{[0, s]} \left(\frac{Z_i - z}{h^{1/d}} \right), \quad s \in [0, 1]^d.$$

These processes, usually called functional increments of the empirical distribution function, have been intensively investigated in the literature (see, e.g., Shorack and Wellner [11], Van der Vaart and Wellner [12], Deheuvels and Mason [5, 3], Einmahl and Mason [7], Mason [9]). A particular domain of investigation of these increments is when their almost sure behavior is studied along a sequence of bandwidths $(h_n)_{n \geq 1}$ satisfying the following conditions:

$$\begin{aligned} (HVE1) \quad & 0 < h_n < 1, \quad h_n \downarrow 0, \quad nh_n \uparrow \infty, \\ (HVE2) \quad & nh_n / \log n \rightarrow c. \end{aligned}$$

Here, $c > 0$ denotes a finite constant. Such conditions on the sequence $(h_n)_{n \geq 1}$ are called Erdős-Rényi conditions, since these two authors have given a pioneering result in this domain (see [?]). Deheuvels and Mason [5] showed that, whenever the $(Z_i)_{i \geq 1}$ are uniformly distributed on $[0, 1]$, and under $(HVE1) - (HVE2)$, the increments $n\Delta_n(z, h, \cdot)/(c \log n)$ have a nonstandard almost sure behaviour. Before citing their result, we need to introduce the following notations. Set $B([0, 1]^d)$ as the cone of all bounded increasing functions g on $[0, 1]^d$ (implicitly with respect to the order \prec), satisfying $g(0) = 0$. We shall endow this cone with the topology spawned by the usual sup-norm $\|g\| := \sup_{s \in [0, 1]^d} |g(s)|$. Define the usually called Chernoff function h as

$$h(x) := \begin{cases} x \log x - x + 1, & \text{for } x > 0; \\ 1, & \text{for } x = 0; \\ \infty, & \text{for } x < 0. \end{cases} \quad (1.2)$$

That function is known to play an important role in the large deviation of Poisson processes on $[0, 1]$ (see, e.g., [8]). Define the following (rate) function on $B([0, 1]^d)$. Whenever $g \in B([0, 1]^d)$ is absolutely continuous with respect to the Lebesgue measure on $[0, 1]^d$, we set

$$I(g) := \int_{[0, 1]^d} h(g'(s)) ds, \quad (1.3)$$

g' denoting (a version of) the derivative of g with respect to the Lebesgue measure. Whenever g fails to be absolutely continuous, we set $I(g) = \infty$. Also define, for any $a > 0$,

$$\Gamma_a := \{g \in B([0, 1]^d), I(g) \leq 1/a\}. \quad (1.4)$$

In a pioneering work, Deheuvels and Mason [5] established the following non standard uniform functional limit law for the $\Delta_n(z, h_n, \cdot)$, when the (Z_i) are uniform on $[0, 1]$.

Theorem 1 (Deheuvels, Mason, 1992) Assume that $d = 1$ and that the $(Z_i)_{i \geq 1}$ are uniformly distributed on $[0, 1]$. Let $0 \leq a < b < 1$ be two real numbers, and let $(h_n)_{n \geq 1}$ be a sequence of positive constants satisfying (HVE1) – (HVE2) for some constant $c > 0$. Then we have almost surely

$$\lim_{n \rightarrow \infty} \sup_{z \in [0, 1 - h_n]} \inf_{g \in \Gamma_c} \left\| \frac{n}{c \log n} \Delta_n(z, h_n, \cdot) - g \right\| = 0,$$

$$\forall g \in \Gamma_c, \lim_{n \rightarrow \infty} \inf_{z \in [0, 1 - h_n]} \left\| \frac{n}{c \log n} \Delta_n(z, h_n, \cdot) - g \right\| = 0.$$

As a corollary, the authors showed that, when the sequence of bandwidth $(h_n)_{n \geq 1}$ satisfies (HVE1) – (HVE2), the Parzen-Rosenblatt kernel density estimator is **not** uniformly strongly consistent. They proved this non-consistency result by making use of some optimisation techniques on Orlicz balls (see Deheuvels and Mason [4]). The aim of the present paper is to provide a generalisation of the former result to the case where the $(Z_i)_{i \geq 1}$ take values in \mathbb{R}^d . This generalisation can be stated as follows.

Theorem 2 Assume that the $(Z_i)_{i \geq 1}$ have a density f satisfying (Hf). Let $H \subset O$ be a compact set with nonempty interior. Let $(h_n)_{n \geq 1}$ be a sequence of positive constants fulfilling (HVE1) and (HVE2). Then we have almost surely

$$(i) \forall z \in H, \forall g \in \Gamma_{cf(z)}, \lim_{n \rightarrow \infty} \inf \left\{ \left\| \Delta_n(z', h_n, \cdot) - g \right\|, z' \in H \right\} = 0, \quad (1.5)$$

$$(ii) \lim_{n \rightarrow \infty} \sup_{z \in H} \inf \left\{ \left\| \Delta_n(z, h_n, \cdot) - g \right\|, g \in \Gamma_{cf(z)} \right\} = 0. \quad (1.6)$$

Denote by $f_n(K, z, h_n)$ the usual kernel density estimator with bandwidth h_n and kernel K . A consequence of Theorem 2 is that, under (HVE1) – (HVE2), $f_n(K, z, h_n)$ is not uniformly consistent (in a strong sense) over (say) an hypercube of \mathbb{R}^d .

Corollary: Let K be a kernel with compact support and bounded variation. Assume (Hf) and (HVE1) – (HVE2). Let $H \subset O$ be a compact with nonempty interior. Then the following event holds with probability one:

$$\exists \epsilon > 0, \exists n_0, \forall n \geq n_0, \sup_{z \in H} |f_n(K, z, h_n) - f(z)| > \epsilon.$$

Proof: The proof follows exactly the lines of Deheuvels and Mason (see [5], Theorem 4.2) and is based on some optimisation results on Orlicz Balls that have been provided in Deheuvels and Mason [4]. \square

From now on, we shall make use of the following notation

$$\Delta_n(z, h_n, s) := \frac{\sum_{i=1}^n 1_{[0, s]} \left(\frac{Z_i - z}{h_n^{1/d}} \right)}{cf(z) \log n}, \quad s \in [0, 1]^d.$$

Remark 1.0.1

Deheuvels and Mason [6] have already given a nonstandard functional limit law for a single increment $\Delta_n(z_0, h_n, \cdot)$ when (HVE2) is replaced by $nh_n/\log \log n \rightarrow c > 0$. Their result is presented in a more general setting, considering the $\Delta_n(z_0, h_n, \cdot)$ as random measures indexed by a class of sets.

The remainder of this paper is organised as follows. In §2 we provide some tools in large deviation theory, which are consequences of results of Arcones [1] and Lynch and Sethuraman [8]. In §3, a uniform large deviation principle for "poissonized" versions of the $\Delta_n(z, h_n, \cdot)$ is established. In §4 and §5, we make use of the just-mentioned uniform large deviation principle to prove Theorem 2.

2 Uniform large deviation principles

The main tool we shall make use of in §4 and §5 is a uniform large deviation principle for a triangular array of compound Poisson processes. We must first remind some usual notions in large deviation theory. Let (E, d) be a metric space. A real function $J : E \rightarrow [0, \infty]$ is said to be a rate function (implicitly for (E, d)) when the sets $\{x \in E : J(x) \leq a\}$, $a \geq 0$, are compact sets of (E, d) . We shall first show that I is a rate function on $(B([0, 1]^d), \|\cdot\|)$ by approximating it by suitably chosen simple rate functions.

2.1 Approximations of I

Given $g \in B([0, 1]^d)$ and a Borel set A , we shall write

$$g(A) := \int_{[0, 1]^d} 1_A dg, \quad (2.1)$$

which is valid as soon as either g or 1_A has bounded variation. For any integer $p \geq 1$ and for each $1 \prec \mathbf{i} \prec 2^p$ set

$$A_{\mathbf{i}}^p := 2^{-p} [\mathbf{i} - 1, \mathbf{i}), \quad (2.2)$$

with the notation $\mathbf{i} - 1 := (i_1 - 1, \dots, i_d - 1)$. Recall that h is given in (1.2), and that λ is the Lebesgue measure on $[0, 1]^d$. The following functions will play the role of approximations of I (given in (1.3)), as $p \rightarrow \infty$:

$$\begin{aligned} I_p(g) &:= \sum_{1 \prec \mathbf{i} \prec 2^p} 2^{-pd} h(2^{pd} g(A_{\mathbf{i}}^p)) \\ &= \sum_{1 \prec \mathbf{i} \prec 2^p} \lambda(A_{\mathbf{i}}^p) h\left(\frac{g(A_{\mathbf{i}}^p)}{\lambda(A_{\mathbf{i}}^p)}\right), \quad g \in B([0, 1]^d). \end{aligned} \quad (2.3)$$

We point out the following properties of the function I .

Proposition 2.1 *For each $g \in B([0, 1]^d)$, we have*

$$\lim_{p \rightarrow \infty} I_p(g) = I(g). \quad (2.4)$$

Moreover, I is a rate function on $(B([0, 1]^d), \|\cdot\|)$.

Proof: Choose $g \in B([0, 1]^d)$ arbitrarily and assume that $I(g) > 0$ (nontrivial case). In a first time, we suppose that g has bounded variation, so that it can be interpreted as a finite measure. Denote by \mathcal{T}_p the σ -algebra of $[0, 1]^d$ spawned by the sets $A_{\mathbf{i}}^p$, $1 \prec \mathbf{i} \prec 2^p$. Clearly, for all $p \geq 1$, the measure g is absolutely continuous with respect to the (trace of the) Lebesgue measure λ on \mathcal{T}_p . Furthermore, the corresponding Radon-Nicodym derivative is given by the following equality.

$$L_p := \frac{dg}{d\lambda} |_{\mathcal{T}_p} = \sum_{1 \prec \mathbf{i} \prec 2^p} 1_{A_{\mathbf{i}}^p} \frac{g(A_{\mathbf{i}}^p)}{\lambda(A_{\mathbf{i}}^p)}. \quad (2.5)$$

Clearly the σ -algebra spawned by the (increasing) sequence $(\mathcal{T}_p)_{p \geq 1}$ is equal to the Borel σ -algebra of $[0, 1]^d$. Assume first that g is absolutely continuous with respect to λ . According to Dacunha-Castelle and Duflo [2], p. 63, the sequence L_p converges $\lambda + g$ almost everywhere to a positive function L satisfying $L = g'$ ($\lambda + g$ almost everywhere). Now select $0 < l < I(g)$ arbitrarily. By definition of I , there exists $\epsilon > 0$ satisfying

$$\int_{\epsilon < L < 1/\epsilon} h(L) d\lambda > l.$$

Since $L_p \rightarrow L$ ($\lambda + g$ almost everywhere as $p \rightarrow \infty$) and since h is continuous, we have

$$\liminf_{p \rightarrow \infty} h(L_p) 1_{\{\epsilon < L_p < 1/\epsilon\}} \geq h(L) 1_{\{\epsilon < L < 1/\epsilon\}} \quad \lambda + g \text{ almost everywhere}$$

Hence by an application of Fatou's lemma,

$$\liminf_{p \rightarrow \infty} \int_{\epsilon < L_p < 1/\epsilon} h(L_p) d\lambda \geq \int_{\epsilon < L < 1/\epsilon} h(L) d\lambda > l.$$

Since $\sup_{p \geq 1} I_p(g) \leq I(g)$ by a straightforward use of Jensen's inequality, and since $l < I(g)$ was chosen arbitrarily, we readily infer that $I_p(g) \rightarrow I(g)$ as $p \rightarrow \infty$. Now assume that $I(g) = \infty$ and that g is not absolutely continuous with respect to λ . According to Dacunha-Castelle and Duflo [2], p. 63, the sequence L_p converges $\lambda + g$ almost everywhere to a positive function L satisfying $(\lambda + g)(\{L = \infty\}) =: \tau > 0$. Define

$$\ell(x) := x^{-1} h(x) = \log(x) - 1 + x^{-1}, \quad x > 0.$$

Clearly, $\ell(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. Now select $l > 0$ arbitrarily, and choose $A > 0$ satisfying

$$\inf_{x > A} \ell(x) > \frac{2l}{\tau}.$$

Since $L_p \rightarrow L$ ($\lambda + g$ almost everywhere as $p \rightarrow \infty$) we have $g(L_p > A) > \tau/2$ for all large p , whence

$$\begin{aligned}
I_p(g) &\geq \int_{L_p \in (A, \infty)} \ell(L_p) L_p d\lambda \\
&= \int_{L_p \in (A, \infty)} \ell(L_p) dg \\
&\geq \frac{2l}{\tau} g(L_p > A) \\
&> l.
\end{aligned} \tag{2.6}$$

We have shown that (2.4) is true for each g with bounded variation. Whenever g has infinite variation, then it can be shown that $I_p(g) \rightarrow \infty$ by a discrete version of the argument that have just been invoked to obtain (2.6). We omit details for sake of brevity.

Since all the functions I_p are $\|\cdot\|$ -continuous and since $I_p(g) \uparrow I(g)$ for all $g \in B([0, 1]^d)$, we conclude that I is lower-semicontinuous for $\|\cdot\|$. Hence, I is a rate function if and only if the set Γ_a is totally bounded for each $a > 0$ (recall (1.4)). Since $x^{-1}h(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, we have, for some constant $M > 0$,

$$|x| \leq |x| 1_{|x| \leq M} + h(x), \tag{2.7}$$

from where we readily infer that

$$\int_{[0, 1]^d} |g'| d\lambda \leq M + 1/a \text{ for each } a > 0 \text{ and } g \in \Gamma_a. \tag{2.8}$$

Applying the Arzela-Ascoli criterion, we conclude that, for each $a > 0$, the closed set Γ_a is totally bounded, which entails that I is a rate function on $(B([0, 1]^d), \|\cdot\|)$. This concludes the proof of Proposition 2.1. \square

2.2 Uniform large deviations in $(B([0, 1]^d), \|\cdot\|)$

We shall now give a definition of a large uniform large deviation principle in the metric space $(B([0, 1]^d), \|\cdot\|)$. In the sequel, $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ will always denote a triangular array of positive numbers satisfying $\max_{i \leq m_n} \epsilon_{n,i} \rightarrow 0$ as $n \rightarrow \infty$. Let $(X_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of random elements on probability space $(\omega, \mathcal{T}', \mathbb{P})$, taking values in $B([0, 1]^d)$. In order to handle carefully the notions of inner and outer probabilities, we shall that each $X_{n,i}$ is a suitable projection mapping from (Ω, \mathcal{T}') to E , where

$$\Omega := \prod_{n=1}^{\infty} \prod_{i=1}^p B([0, 1]^d), \quad \mathcal{T}' := \bigotimes_{n=1}^{\infty} \bigotimes_{i=1}^p \mathcal{T},$$

and \mathcal{T} is the Borel σ -algebra of $(B([0, 1]^d), \|\cdot\|)$. From now on, outer and inner probabilities \mathbb{P}^* and \mathbb{P}_* are understood with (Ω, \mathcal{T}) as the underlying probability space. We say that $(X_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the Uniform Large Deviation Principle (ULDP) for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for a rate function J whenever the two following conditions hold.

- For any $\|\cdot\|$ -open set $O \subset B([0, 1]^d)$ we have

$$\liminf_{n \rightarrow \infty} \min_{i \leq m_n} \epsilon_{n,i} \log(\mathbb{P}_*(X_{n,i}(\cdot) \in O)) \geq -J(O). \quad (2.9)$$

- For any $\|\cdot\|$ -closed set $F \subset B([0, 1]^d)$ we have

$$\limsup_{n \rightarrow \infty} \max_{i \leq m_n} \epsilon_{n,i} \log(\mathbb{P}^*(X_{n,i}(\cdot) \in F)) \leq -J(F). \quad (2.10)$$

Remark 2.2.1

The same definition holds for triangular arrays of random variables taking values in \mathbb{R}^p , $p \geq 1$. The norm $\|\cdot\|$ can then be replaced by any norm.

Arcones [1] provided a powerful tool to establish Large Deviation Principles for sequences of bounded stochastic processes. Some verifications lead to the conclusion that the just-mentioned tool can be used in our context. Recall that the sets A_i^p have been defined by (2.2). Consider the following finite grid, for $p \geq 1$:

$$s_{i,p} := 2^{-p}(\mathbf{i} - 1), \quad 1 \prec \mathbf{i} \prec 2^p. \quad (2.11)$$

Given, $p \geq 1$ and $g \in B([0, 1]^d)$, we write

$$g^{(p)} = \sum_{1 \prec \mathbf{i} \prec 2^p} 1_{A_i^p} g(s_{i,p}).$$

Proposition 2.2 *Let $(X_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of random elements taking values in $(B([0, 1]^d))$ almost surely, and let $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of positive real numbers. Assume that the following conditions are satisfied.*

1. *The triangular array of stochastic process $(X_{n,i}^{(p)})_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the rate function I_p on $(B([0, 1]^d), \|\cdot\|)$.*
2. *For each $\tau > 0$ and $M > 0$ there exists $p \geq 1$ satisfying*

$$\limsup_{n \rightarrow \infty} \max_{i \leq m_n} \epsilon_{n,i} \log \left(\mathbb{P}^* \left(\max_{1 \prec \mathbf{i} \prec 2^p} \sup_{s \in A_i^p} |X_{n,i}(t) - X_{n,i}(s_{\mathbf{i}}^p)| \geq \tau \right) \right) \leq -M.$$

Then $(X_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function.

$$J(g) := \sup_{p \geq 1} I_p(g^{(p)}), \quad g \in B([0, 1]^d).$$

Proof: The proof follows exactly the same lines as in the proof of Theorem 3.1 of Arcones [1]. Using these arguments in our context remains possible since the cone $B([0, 1]^d)$ is a closed subset of $L^\infty([0, 1]^d)$ for the usual sup norm $\|\cdot\|$. We avoid writing the proof for sake of brevity. \square

Another tool we shall make an intensive use of is a ULDP for random vectors with mutually independent coordinates.

Proposition 2.3 *Let $(X_{n,i})_{n \geq 1, 1 \leq i \leq m_n}$ and $(Y_{n,i})_{n \geq 1, 1 \leq i \leq m_n}$ be two triangular arrays of random vectors taking values in \mathbb{R}^d and $\mathbb{R}^{d'}$ respectively, and satisfying $X_{n,i} \perp Y_{n,i}$ for each $n \geq 1, 1 \leq i \leq m_n$. Assume that both $(X_{n,i})_{n \geq 1, 1 \leq i \leq m_n}$ and $(Y_{n,i})_{n \geq 1, 1 \leq i \leq m_n}$ satisfy the ULDP for a triangular array $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for two rate functions J_1 and J_2 respectively. Then the triangular array $(X_{n,i}, Y_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function.*

$$J(z_1, z_2) := J_1(z_1) + J_2(z_2), \quad z_1 \in \mathbb{R}^d, \quad z_2 \in \mathbb{R}^{d'}.$$

Proof: The proof follows the same lines as Lemma 2.6 and Corollary 2.9 in Lynch and Sethuraman [8]. In the just-mentioned article, the authors make use of the notions of Weak Large Deviation Principle and of LD-tightness for sequences of random variables in a Polish space. These notions can be easily extended to the frame of triangular arrays of random variables. \square

The following proposition is nothing else than the contraction principle in the framework of ULDP (see, e.g., [1], Theorem 2.1 for the most general version of that principle).

Proposition 2.4 *Let $(X_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular arrays of \mathbb{R}^p valued random vectors satisfying the ULDP for a triangular array $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for a rate function J . Let \mathcal{R} be a continuous mapping from \mathbb{R}^d to $(B([0, 1]^d), \|\cdot\|)$. Then $(\mathcal{R}(X_{n,i}))_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function.*

$$J_{\mathcal{R}}(g) := \inf\{J(x), \mathcal{R}(x) = g\}, \quad g \in B([0, 1]^d),$$

with the convention $\inf \emptyset = \infty$.

Proof: Straightforward. \square

The following proposition shall be useful in our the proof of our Lemma 3.1.

Proposition 2.5 *Let $(X_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of real random variables and let $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ be a triangular array of positive real numbers. Assume that there exists a strictly convex positive function J on \mathbb{R} and a real number μ such that $J(\mu) = 0$ and*

$$\forall a > \mu, \quad \lim_{n \rightarrow \infty} \max_{i \leq m_n} \left| \epsilon_{n,i} \log(\mathbb{P}(X_{n,i} \geq a)) - J(a) \right| = 0, \quad (2.12)$$

$$\forall a < \mu, \quad \lim_{n \rightarrow \infty} \max_{i \leq m_n} \left| \epsilon_{n,i} \log(\mathbb{P}(X_{n,i} \leq a)) - J(a) \right| = 0. \quad (2.13)$$

Then $(X_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for J .

Proof: The proof is routine calculus. \square

3 A ULDP for poissonised versions of the $\Delta_n(z, h_n, \cdot)$

Define the following process, for each integer $n \geq 1$.

$$\Delta\Pi_n(z, h_n, s) := \frac{\sum_{i=1}^{\eta_n} 1_{[0,s]} \left(\frac{Z_i - z}{h_n^{1/d}} \right)}{cf(z) \log n}, \quad s \in [0, 1]^d. \quad (3.1)$$

Here η_n is a Poisson random variable independent of $(Z_i)_{i \geq 1}$, with expectation n . These "poissonized" versions of the processes $\Delta_n(z, h_n, \cdot)$ can be identified to random (Poisson) measures by the following relation

$$\Delta\Pi_n(z, h_n, A) := \int_{[0,1]^d} 1_A(s) d\Delta\Pi_n(z, h_n, s), \quad A \text{ Borel}. \quad (3.2)$$

The key of our proof of Theorem 2 is the following ULDP.

Proposition 3.1 *Let $(z_{i,n})_{n \geq 1, 1 \leq i \leq m_n}$ be a triangular array of elements of H . Under the assumptions of Theorem 2, the triangular array of processes $(\Delta\Pi_n(z_{i,n}, h_n, \cdot))_{n \geq 1, 1 \leq i \leq m_n}$ satisfies the ULDP in $(B([0, 1]^d), \|\cdot\|)$ for the rate function I and for the following triangular array*

$$\epsilon_{n,i} := \frac{1}{cf(z_{i,n}) \log n}, \quad n \geq 1, 1 \leq i \leq m_n. \quad (3.3)$$

Remark 3.0.2

Proposition 3.1 is true whatever the constant $c > 0$ appearing in assumption (HVE1). This remark will show up to be useful in Lemma 5.2 in §5.

Proof: To prove proposition 3.1, we shall make use of Proposition 2.2. We hence have to check conditions 1, 2 and 3 of the just-mentioned proposition. This will be achieved through several lemmas.

3.1 A preliminary lemma

Recall notation (2.1). To check condition 2 of Proposition 2.2, we need first to establish the following lemma.

Lemma 3.1 *Assume that the hypothesis of Theorem 2 are satisfied. Then, for each $p \geq 1$ and for each $1 \prec \mathbf{i}_0 \prec 2^p$, the triangular array of random variables $(\Delta\Pi_n(z_{i,n}, h_n, A_{\mathbf{i}_0}^p))_{n \geq 1, 1 \leq i \leq m_n}$ satisfies the ULDP in $[0, \infty)$ for the triangular array $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function:*

$$\tilde{I}_p(x) := 2^{-pd} h \left(\frac{x}{2^{-pd}} \right) = \lambda(A_{\mathbf{i}_0}^p) h \left(\frac{x}{\lambda(A_{\mathbf{i}_0}^p)} \right), \quad x \geq 0. \quad (3.4)$$

Proof: Fix once for all $p \geq 1$ and $1 \prec \mathbf{i}_0 \prec 2^d$. We shall make use of Proposition 2.5, with $J := \tilde{I}_p$ and $\mu := 2^{-pd}$. We give details only for the proof of (2.12), as proving (2.13) is very similar. Fix $a > 2^{-pd}$. For each integers $n \geq 1$ and $1 \leq i \leq m_n$, we set (recall (3.2))

$$\begin{aligned} V_{i,n,\mathbf{i}_0} &:= cf(z_{i,n})(\log n) \Delta \Pi_n(z_{i,n}, h_n, A_{\mathbf{i}_0}^p), \\ p_{i,n,\mathbf{i}_0} &:= \mathbb{P} \left(Z_1 \in z_{i,n} + h_n^{1/d} A_{\mathbf{i}_0}^p \right). \end{aligned}$$

Clearly V_{i,n,\mathbf{i}_0} is a Poisson random variable with expectation np_{i,n,\mathbf{i}_0} . Since the density f satisfies (Hf) and since $\lambda(A_{\mathbf{i}_0}^p) = 2^{-pd}$, we have

$$\lim_{n \rightarrow \infty} \max_{1 \leq i \leq m_n} \left| \frac{p_{i,n,\mathbf{i}_0}}{f(z_{i,n})2^{-pd}h_n} - 1 \right| = 0. \quad (3.5)$$

Hence according to (HVE2) we have, ultimately as $n \rightarrow \infty$,

$$\min_{1 \leq i \leq m_n} \frac{acf(z_{i,n}) \log n}{np_{i,n,\mathbf{i}_0}} > 1. \quad (3.6)$$

We then make use of Chernoff's inequality for Poisson random variables to get, for all large n (satisfying (3.6)) and for all $1 \leq i \leq m_n$,

$$\begin{aligned} \mathbb{P} \left(\Delta \Pi_n(z_{i,n}, h_n, A_{\mathbf{i}_0}^p) \geq a \right) &= \mathbb{P} (V_{i,n,\mathbf{i}_0} \geq acf(z_{i,n}) \log n) \\ &\leq \exp \left(-np_{i,n,\mathbf{i}_0} h \left(\frac{acf(z_{i,n}) \log n}{np_{i,n,\mathbf{i}_0}} \right) \right). \end{aligned} \quad (3.7)$$

But (3.7) in combination with (3.5) entails

$$\limsup_{n \rightarrow \infty} \max_{1 \leq i \leq m_n} \frac{p_{i,n,\mathbf{i}_0}}{f(z_{i,n})h_n} h \left(\frac{acf(z_{i,n}) \log n}{np_{i,n,\mathbf{i}_0}} \right) \leq 2^{-pd} h \left(\frac{a}{2^{-pd}} \right), \quad (3.8)$$

which, together with (3.7) leads to

$$\limsup_{n \rightarrow \infty} \max_{1 \leq i \leq m_n} \epsilon_{n,i} \log \left(\mathbb{P} \left(\Delta \Pi_n(z_{i,n}, h_n, A_{\mathbf{i}_0}^p) \geq a \right) \right) \leq -\tilde{I}_p(a). \quad (3.9)$$

Now select $y > a$ arbitrarily. If we could show that

$$\liminf_{n \rightarrow \infty} \min_{1 \leq i \leq m_n} \epsilon_{n,i} \log \left(\mathbb{P} \left(\Delta \Pi_n(z_{i,n}, h_n, A_{\mathbf{i}_0}^p) \geq a \right) \right) \geq -\tilde{I}_p(y),$$

then, as $y > a$ was chosen arbitrarily, and since \tilde{I}_p is increasing on $[a, \infty)$, we should be able to conclude the proof of (2.12) with $J = \tilde{I}_p$. Now set $\phi(t) := \exp(\exp(t) - 1)$, $t \in \mathbb{R}$ and notice that $h(z) = \max_{u \in \mathbb{R}} zu - \log(\phi(u))$ for each $z > 0$. Set $u_0 := \log(2^{pd}y)$, so as

$$h(2^{pd}y) = 2^{pd}yu_0 - \log(\phi(u_0)). \quad (3.10)$$

Denote by F the distribution function of a Poisson random variable with expectation 1, and define F_0 by

$$dF_0(x) := \phi(u_0)^{-1} \exp(u_0 x) dF(x). \quad (3.11)$$

Let $"*"$ be the convolution operator for infinitely divisible laws and notice that, for each $L > 0$, we have

$$dF_0^{*L}(\cdot) = \phi(u_0)^{-L} \exp(u_0 \cdot) dF^{*L}(\cdot), \quad (3.12)$$

$$\mathbb{E}_{F_0^{*L}}(X) = 2^{pd} L y, \quad (3.13)$$

$$\text{Var}_{F_0^{*L}}(X) = L \text{Var}_{F_0}(X) \quad (3.14)$$

Here we have written $\mathbb{E}_F(X)$ as the expectation of a random variable with distribution F . Now fix $\delta > 0$ satisfying $[y - \delta, y + \delta] \subset [a, \infty[$ arbitrarily. Obviously, $F^{*np_{i,n}, i_0}$ is the distribution function of $cf(z_{i,n})(\log n) \Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p)$, whence

$$\begin{aligned} & \mathbb{P}(\Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p) \geq a) \\ & \geq \mathbb{P}(\Delta \Pi_n(z_{i,n}, h_n, A_{i_0}^p) \in [y - \delta, y + \delta]) \\ & = \int_{\frac{x}{cf(z_{i,n}) \log n} \in [y - \delta, y + \delta]} dF^{*np_{i,n}, i_0}(x) \\ & \geq \exp(-u_0(y + \delta)cf(z_{i,n}) \log n) \times \int_{\frac{x}{cf(z_{i,n}) \log n} \in [y - \delta, y + \delta]} \exp(u_0 x) dF^{*np_{i,n}, i_0}(x) \\ & \geq \exp(-cf(z_{i,n})(\log n)u_0(y + \delta) + np_{i,n}, i_0 \log(\phi(u_0))) \\ & \quad \times \int_{\frac{x}{cf(z_{i,n}) \log n} \in [y - \delta, y + \delta]} dF_0^{*np_{i,n}, i_0}(x) \\ & := a_{i,n, i_0, \delta} \times b_{i,n, i_0, \delta}. \end{aligned} \quad (3.15)$$

Here (3.15) is a consequence of (3.12), with $L := np_{i,n}, i_0$. Now let $n \geq 1$ be an integer large enough to fulfill (recall (3.5))

$$\max_{1 \leq i \leq m_n} \left| \frac{np_{i,n}, i_0}{2^{-pd} cf(z_{i,n}) \log n} - 1 \right| \leq u_0 \log(\phi(u_0))^{-1} \delta, \quad (3.16)$$

which enables us to write the following chain of inequalities.

$$\begin{aligned} & cf(z_{i,n})(\log n)u_0(y + \delta) - np_{i,n}, i_0 \log(\phi(u_0)) \\ & \leq 2^{-pd}(y + \delta)cf(z_{i,n}) \log n (u_0 2^{pd} - \log(\phi(u_0)) + u_0 \delta) \\ & \leq 2^{-pd} cf(z_{i,n}) \log n (h(2^{pd} y) + u_0(2^{pd} + 1)\delta) \\ & = cf(z_{i,n}) \log n \left(\tilde{I}_p(y) + 2^{-pd} (2^{pd} + 1) u_0 \delta \right) \\ & \leq cf(z_{i,n}) \log n \left(\tilde{I}_p(y) + 2u_0 \delta \right). \end{aligned} \quad (3.17)$$

Therefore we have, for all large n and for all $1 \leq i \leq m_n$,

$$a_{i,n,\mathbf{i}_0,\delta} \geq \exp \left(-cf(z_{i,n}) \log n \left(\tilde{I}_p(y) + 2u_0\delta \right) \right), \quad (3.18)$$

where $u_0 = \log(2^{pd}y)$ depends on $y > a$ only. It remains to show that

$$\lim_{n \rightarrow \infty} \min_{1 \leq i \leq m_n} b_{i,n,\mathbf{i}_0,\delta} = 1. \quad (3.19)$$

Consider n large enough to fulfill (recall (3.5))

$$\frac{y - \delta}{y + 2^{-pd}\delta} < \min_{1 \leq i \leq m_n} \frac{np_{i,n,\mathbf{i}_0}}{2^{-pd}cf(z_{i,n}) \log n} \leq \max_{1 \leq i \leq m_n} \frac{np_{i,n,\mathbf{i}_0}}{2^{-pd}cf(z_{i,n}) \log n} < \frac{y + \delta}{y - 2^{-pd}\delta},$$

so as, for all $1 \leq i \leq m_n$,

$$\frac{np_{i,n,\mathbf{i}_0}}{2^{-pd}cf(z_{i,n}) \log n} \times [y - 2^{-pd}\delta, y + 2^{-pd}\delta] \subset]y - \delta, y + \delta[, \quad (3.20)$$

and hence

$$b_{i,n,\mathbf{i}_0,\delta} \geq \frac{x}{\frac{x}{np_{i,n,\mathbf{i}_0}} \in [2^{pd}y - \delta, 2^{pd}y + \delta]} \int_{\frac{x}{np_{i,n,\mathbf{i}_0}} \in [2^{pd}y - \delta, 2^{pd}y + \delta]} dF_0^{*np_{i,n,\mathbf{i}_0}}(x).$$

Recalling (3.13) and (3.14) we get, by the Bienaymé-Tchebychev inequality,

$$1 - b_{i,n,\mathbf{i}_0,\delta} \leq \frac{\text{Var}_{F_0}(X)}{\delta np_{i,n,\mathbf{i}_0}}. \quad (3.21)$$

By assumption (Hf) we infer that the $h_n^{-1}p_{i,n,\mathbf{i}_0}$ are bounded away from zero, from where (3.15) follows. Then (3.15), (3.18) and (3.19) entail

$$\liminf_{n \rightarrow \infty} \min_{1 \leq i \leq m_n} \epsilon_{n,i} \log \left(\mathbb{P} \left(\Delta \Pi_n(z_{i,n}, h_n, A_{\mathbf{i}_0}^p) \geq a \right) \right) \geq -\tilde{I}_p(y) - 2u_0\delta. \quad (3.22)$$

Assertion (2.12) is then proved by combining (3.9) with (3.22), as $\delta > 0$ is arbitrary. \square

3.2 Verification of condition 2 of Proposition 2.2

For $n \geq 1$ and $1 \leq i \leq m_n$, define the following $\mathbb{R}^{2^{pd}}$ valued random vector:

$$\begin{aligned} X_{n,i} &:= (X_{\mathbf{i}_0,n,i})_{1 \prec \mathbf{i}_0 \prec 2^p} \\ &:= (\Delta \Pi_n(z_{i,n}, h_n, A_{\mathbf{i}_0}^p))_{1 \prec \mathbf{i}_0 \prec 2^p}. \end{aligned}$$

Notice that the random variables $X_{i_0,n,i}$, $1 \prec \mathbf{i}_0 \prec 2^p$ are mutually independent for fixed $n \geq 1$ and $1 \leq i \leq m_n$ by usual properties of Poisson random measures.

Hence, by Lemma 3.1 together with Proposition 2.3 we deduce that the triangular array $(X_{n,i})_{n \geq 1, i \leq m_n}$ satisfies the ULDP with $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and with the following rate function.

$$I'_p(x) := \sum_{1 \prec \mathbf{i} \prec 2^p} 2^{-pd} h\left(\frac{x_{\mathbf{i}}}{2^{-pd}}\right), \quad x \in [0, \infty)^{2^{pd}}. \quad (3.23)$$

Here we have written $x := (x_{\mathbf{i}})_{1 \prec \mathbf{i} \prec 2^p}$. We now define the following mappings from $[0, \infty)^{2^{pd}}$ to $(B([0, 1]^d))$

$$\begin{aligned} \mathcal{R}_p(x) : [0, 1]^d &\mapsto [0, \infty) \\ s &\rightarrow \sum_{A_{\mathbf{i}}^p \subset [0, s]} x_{\mathbf{i}}. \end{aligned}$$

Denote by $[x]$ the integer part of a real number x ($[x] \leq x < [x] + 1$), and write $[s] := ([s_1], \dots, [s_d])$ for any $s = (s_1, \dots, s_d) \in \mathbb{R}^d$. We point out that with probability one (recall the notations of Proposition 2.2)

$$\begin{aligned} \mathcal{R}_p(X_{n,i})(s) &= \Delta \Pi_n(z_{i,n}, h_n, 2^{-p}[2^p s]) \\ &= \Delta \Pi_n(z_{i,n}, h_n, s)^{(p)}, \quad s \in [0, 1]^d. \end{aligned}$$

For fixed $p \geq 1$, we make use of the contraction principle (Proposition 2.4) to conclude that $(\mathcal{R}_p(X_{n,i}))_{n \geq 1, i \leq m_n}$ satisfies the ULDP for $(\epsilon_{n,i})_{n \geq 1, i \leq m_n}$ and for the following rate function.

$$\bar{I}_p(g) := \inf \left\{ I'_p(x), \quad x \in [0, \infty)^{2^{pd}}, \quad \mathcal{R}_p(x) = g \right\}, \quad g \in B([0, 1]^d), \quad (3.24)$$

with the convention $\inf \emptyset = \infty$. Obviously, the set appearing in (3.24) is non void if and only if g is the cumulative distribution function of a purely atomic measure with atoms belonging to the grid $\{s_{\mathbf{i},p}, \quad 1 \prec \mathbf{i} \prec 2^p\}$. In that case we have

$$\bar{I}_p(g) = \sum_{1 \prec \mathbf{i} \prec 2^p} 2^{-pd} h\left(\frac{g(A_{\mathbf{i}}^p)}{2^{-pd}}\right) = I_p(g).$$

Here, we have identified g to a positive finite measure on $[0, 1]^d$ (recall (2.1)). Assumption 2 of Proposition 2.2 is then satisfied.

3.3 Verification of condition 3 of Proposition 2.2

Fix $\tau > 0$ and $M > 0$. We have to prove that, provided that p is large enough,

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \max_{1 \leq i \leq m_n} \epsilon_{n,i} \\ &\log \left(\mathbb{P} \left(\max_{1 \prec \mathbf{i} \prec 2^p} \sup_{s \in A_{\mathbf{i}}^p} \left| \Delta \Pi_n(z_{i,n}, h_n, s) - \Delta \Pi_n(z_{i,n}, h_n, 2^{-p}(\mathbf{i} - 1)) \right| \geq \tau \right) \right) \\ &\leq -M. \end{aligned} \quad (3.25)$$

For fixed $p \geq 1$, $n \geq 1$, $1 \leq i \leq m_n$, a rough upper bound gives

$$\begin{aligned}
& \mathbb{P} \left(\max_{1 \prec \mathbf{i} \prec 2^p} \sup_{s \in A_1^p} \left| \Delta \Pi_n(z_{i,n}, h_n, s) - \Delta \Pi_n(z_{i,n}, h_n, 2^{-p}(\mathbf{i} - 1)) \right| \geq \tau \right) \\
& \leq 2^{pd} \max_{1 \prec \mathbf{i} \prec 2^p} \mathbb{P} \left(\sup_{\substack{2^{-p}(\mathbf{i}-1) \prec s \\ \prec 2^{-p}\mathbf{i}}} \left| \Delta \Pi_n(z_{i,n}, h_n, s) - \Delta \Pi_n(z_{i,n}, h_n, 2^{-p}(\mathbf{i} - 1)) \right| \geq \tau \right) \\
& \leq \mathbb{P}(\Delta \Pi_n(z_{i,n}, h_n, 2^{-p}\mathbf{i}) - \Delta \Pi_n(z_{i,n}, h_n, 2^{-p}(\mathbf{i} - 1)) \geq \tau) \\
& =: \mathbb{P}_{i,n,\mathbf{i},p}.
\end{aligned} \tag{3.26}$$

We shall now write

$$\begin{aligned}
W_{i,n,\mathbf{i},p} &:= cf(z_{i,n}) \log n (\Delta \Pi_n(z_{i,n}, h_n, 2^{-p}\mathbf{i}) - \Delta \Pi_n(z_{i,n}, h_n, 2^{-p}(\mathbf{i} - 1))), \\
\mu_{i,n,\mathbf{i},p} &:= \mathbb{P} \left(\frac{Z_1 - z_{i,n}}{h_n^{1/d}} \in [0, 2^{-p}\mathbf{i}) - [0, 2^{-p}(\mathbf{i} - 1)) \right), \text{ and} \\
\nu_{i,p} &:= \lambda([0, 2^{-p}\mathbf{i}) - [0, 2^{-p}(\mathbf{i} - 1))) \leq d2^{-p}.
\end{aligned} \tag{3.27}$$

Clearly, $W_{i,n,\mathbf{i},p}$ is a Poisson random variable with expectation $n\mu_{i,n,\mathbf{i},p}$. Moreover, by assumption (Hf) we have

$$\lim_{n \rightarrow \infty} \min_{\substack{1 \leq i \leq m_n, \\ 1 \prec \mathbf{i} \prec 2^p}} \frac{cf(z_{i,n})(\log n)\nu_{i,p}}{n\mu_{i,n,\mathbf{i},p}} = 1. \tag{3.28}$$

Recall that $x^{-1}h(x) \rightarrow \infty$ as $x \rightarrow \infty$. We can then choose $A_{M,\tau} > 1$ large enough to satisfy

$$\inf_{x \geq A_{M,\tau}} \frac{h(x)}{x} > \frac{8M}{\tau}. \tag{3.29}$$

By (3.27) we can choose p large enough to fulfill

$$\min_{1 \prec \mathbf{i} \prec 2^p} \frac{\tau}{2\nu_{i,p}} > A_{\tau,M}. \tag{3.30}$$

Assertion (3.28) together with (3.30) leads to the following inequality, for all large n , for all $1 \leq i \leq m_n$ and for all $1 \prec \mathbf{i} \prec 2^p$.

$$\frac{cf(z_{i,n})\tau \log n}{n\mu_{i,n,\mathbf{i},p}} \geq \frac{\tau}{2\nu_{i,p}} > A_{\tau,M} > 1. \tag{3.31}$$

Applying Chernoff's inequality to the Poisson random variables $W_{i,n,\mathbf{i},p}$ we get, for all large n and for all $1 \leq i \leq m_n$,

$$\begin{aligned}
\mathbb{P}_{i,n,\mathbf{i},p} &= \mathbb{P}(W_{i,n,\mathbf{i},p} \geq \tau cf(z_{i,n}) \log n) \\
&\leq \exp \left(-n\mu_{i,n,\mathbf{i},p} h \left(\frac{cf(z_{i,n})\tau \log n}{n\mu_{i,n,\mathbf{i},p}} \right) \right).
\end{aligned}$$

Therefore, recalling (3.28) and (3.31), the following inequality holds for all large n , for all $1 \leq i \leq m_n$ and for all $1 \prec \mathbf{i} \prec 2^p$.

$$\begin{aligned} \mathbb{P}_{i,n,\mathbf{i},p} &\leq \exp \left(-\frac{1}{2} cf(z_{i,n}) \nu_{\mathbf{i},p}(\log n) h \left(\frac{\tau}{2\nu_{\mathbf{i},p}} \right) \right) \\ &\leq \exp(-cf(z_{i,n}) 2M \log n). \end{aligned} \quad (3.32)$$

Here, (3.32) is a consequence of (3.30). By combining (3.32) with and (3.26) we get, for all large n and for each $1 \leq i \leq m_n$,

$$\begin{aligned} &\mathbb{P} \left(\max_{1 \prec \mathbf{i} \prec 2^p} \sup_{s \in A_{\mathbf{i}}^p} \left| \Delta \Pi_n(z_{i,n}, h_n, s) - \Delta \Pi_n(z_{i,n}, h_n, 2^{-p}(\mathbf{i} - 1)) \right| \geq \tau \right) \\ &\leq \exp(-2Mcf(z_{i,n}) \log n + \log(2^{pd})), \end{aligned}$$

which proves (3.25) and shows that condition 3 of Proposition 2.2 is satisfied, as f is bounded away from zero on H . We can now make use of the just-mentioned proposition in combination with Proposition 2.1 to conclude the proof of Proposition 3.1. \square

4 Proof of part (i) of Theorem 2

Denote by $\text{Int}(H)$ the interior of H , and fix $z \in \text{Int}(H)$, $g \in \Gamma_{cf(z)}$, and $\epsilon > 0$. We set

$$g^\epsilon := \{g' \in B([0, 1]^d), \|g' - g\| < \epsilon\}. \quad (4.1)$$

By lower semi continuity of I in $(B([0, 1]^d), \|\cdot\|)$ (recall Proposition 2.1), there exists $\alpha_1 > 0$ satisfying

$$I(g^\epsilon) = \frac{1 - 3\alpha_1}{cf(z)}. \quad (4.2)$$

Now choose an hypercube with nonempty interior $H' := [a_1, b_1] \times \dots \times [a_p, b_p]$ fulfilling $H' \subset H$, $\mathbb{P}(Z_1 \in H') \leq 1/2$ and

$$\inf_{z' \in H'} \frac{f(z')}{f(z)} > \frac{1 - 2\alpha_1}{1 - \alpha_1}. \quad (4.3)$$

Such a choice is possible since H has a nonempty interior by assumption. We now divide H' into disjoint hypercubes $z_{i,n} + h_n^{1/d}[0, 1]^d$, $1 \leq i \leq m_n$, where m_n is the maximal number of disjoint hypercubes we can construct without violating

$$\bigcup_{i=1}^{m_n} \{z_{i,n} + h_n^{1/d}[0, 1]^d\} \subset H'. \quad (4.4)$$

Notice that, as $n \rightarrow \infty$,

$$m_n = h_n^{-1+o(1)} = n^{(1+o(1))}. \quad (4.5)$$

Now recall (3.1). By making use of a well-known "poissonization" technique (see, e.g., Mason [10], Fact 6), we get the following upper bound for all large n .

$$\begin{aligned} & \mathbb{P} \left(\bigcap_{z' \in H} \{ \Delta_n(z', h_n, \cdot) \notin g^\epsilon \} \right) \\ & \leq \mathbb{P} \left(\bigcap_{i=1}^{m_n} \{ \Delta_n(z_{i,n}, h_n, \cdot) \notin g^\epsilon \} \right) \\ & \leq 2 \mathbb{P} \left(\bigcap_{i=1}^{m_n} \{ \Delta \Pi_n(z_{i,n}, h_n, \cdot) \notin g^\epsilon \} \right) \end{aligned} \quad (4.6)$$

$$= 2 \prod_{i=1}^{m_n} (1 - \mathbb{P}(\Delta \Pi_n(z_{i,n}, h_n, \cdot) \in g^\epsilon)) \quad (4.7)$$

$$\leq 2 \exp \left(-m_n \min_{1 \leq i \leq m_n} \mathbb{P}(\Delta \Pi_n(z_{i,n}, h_n, \cdot) \in g^\epsilon) \right) \quad (4.8)$$

The transition between (4.6) and (4.7) is a classical property of Poisson random measures, while inequality (4.8) is a consequence of $1 - u \leq \exp(-u)$, $u \geq 0$. We now make use of Proposition 3.1 (with the open ball g^ϵ) to get, for all large n (recall (4.2)),

$$\begin{aligned} \mathbb{P} \left(\bigcap_{z' \in H} \{ \Delta_n(z', h_n, \cdot) \notin g^\epsilon \} \right) & \leq 2 \exp \left(-m_n \min_{1 \leq i \leq m_n} n^{-\frac{f(z_{i,n})}{f(z)}(1-2\alpha_1)} \right) \\ & \leq \exp(-n^{\alpha_1}), \end{aligned}$$

which is a consequence of (4.3) and (4.5). Hence we conclude by the Borel-Cantelli lemma that, almost surely,

$$\lim_{n \rightarrow \infty} \inf \{ \| \Delta_n(z', h_n, \cdot) - g \|, z' \in H \} \leq \epsilon.$$

As $\epsilon > 0$ was chosen arbitrarily, the proof of part (i) of Theorem 2 is concluded for each $z \in \text{Int}(H)$. Now the case where $z \in H$ does not belong to $\text{Int}(H)$ is treated by making use of the following argument: for each $z_1 \in H$, $g_1 \in \Gamma_{cf(z_1)}$ and $\epsilon > 0$, there exists $z_2 \in \text{Int}(H)$ and $g_2 \in \Gamma_{cf(z_2)}$ satisfying $\| g_1 - g_2 \| < \epsilon$. Such an argument is valid by (Hf) and by Lemma 5.1 (see below). \square

5 Proof of part (ii) of Theorem 2

We shall make use of somewhat usual blocking arguments along the following subsequence $n_k := \lfloor \exp(k/\log k) \rfloor$, $k \geq 3$ and its associated blocks $N_k := \{n_{k-1} + 1, \dots, n_k\}$. Given $A \subset B([0, 1]^d)$ and $\epsilon > 0$ we shall write

$$A^\epsilon := \left\{ g \in B([0, 1]^d), \inf_{g' \in A} \| g - g' \| < \epsilon \right\}. \quad (5.1)$$

The following lemma shall come in handy.

Lemma 5.1 *For any $\epsilon > 0$ and $L > 0$ there exists $\eta > 0$ satisfying, for each, $L' \in [(1 + \eta)^{-1}L, L]$, $\Gamma_{L'} \subset \Gamma_L^\epsilon$.*

Proof: The proof is routine analysis. \square

Now fix $\epsilon > 0$. Since I is lower-semi continuous on $(B([0, 1]^d), \|\cdot\|)$ (recall Proposition 2.1) we deduce that, given $z \in H$, there exists $\alpha_z > 0$ satisfying

$$I\left(B([0, 1]^d) - \Gamma_{cf(z)}^\epsilon\right) = \frac{1 + 3\alpha_z}{cf(z)}. \quad (5.2)$$

By (Hf) and Lemma 5.1 we can construct an hypercube H_z with nonempty interior satisfying the following conditions.

$$z \in H_z, \quad H_z \subset O, \quad (5.3)$$

$$\inf_{z_1, z_2 \in H_z} \frac{f(z_1)}{f(z_2)} \geq \frac{1 + \alpha_z}{1 + 2\alpha_z}, \quad (5.4)$$

$$\bigcup_{z' \in H_z} \Gamma_{cf(z')} \subset \Gamma_{cf(z)}^\epsilon, \quad (5.5)$$

$$\mathbb{P}\left(Z_1 \in \bigcup_{z \in H_z} \left\{z + [0, h_{n_k}^{1/d}]^d\right\}\right) \leq 1/2. \quad (5.6)$$

The compact set H is included in the union of the interiors of H_z , $z \in H$, from where we can extract a finite union, noted as

$$H \subset \bigcup_{l=1}^L \text{Int} H_{z_l} \subset \bigcup_{l=1}^L H_{z_l} \subset O. \quad (5.7)$$

Our problem is now reduced to showing that, for fixed $l = 1, \dots, L$,

$$\limsup_{n \rightarrow \infty} \sup_{z \in H_{z_l}} \inf_{g \in \Gamma_{cf(z_l)}} \|\Delta_n(z, h_n, \cdot) - g\| \leq 10\epsilon \text{ almost surely}. \quad (5.8)$$

We now fix $1 \leq l \leq L$, and we write $H_{z_l} =: [a_1, b_1] \times \dots \times [a_d, b_d]$. We now introduce a parameter $\delta > 0$ that will be chosen in function of ϵ in the sequel. For each $k \geq 1$, we cover H_{z_l} by hypercubes

$$H_{z_l} \subset \bigcup_{1 \leq i \leq m_{n_k}} C_{i, n_k} \subset O, \quad (5.9)$$

with

$$C_{i, n_k} := z_{i, n_k} + [0, (\delta h_{n_k})^{1/d}]^d, \quad k \geq 1, \quad 1 \leq i \leq m_{n_k} \text{ and} \\ m_{n_k} := \prod_{p=1}^d \left(\left\lceil \frac{b_p - a_p}{(\delta h_{n_k})^{1/d}} \right\rceil + 1 \right). \quad (5.10)$$

Now define, for each $k \geq 1$, $n \in N_k$, $z \in H$,

$$\mathcal{H}_n(z, s) := \frac{1}{c \log n_k} \sum_{i=1}^n 1_{[0, s)} \left(\frac{Z_i - z}{h_{n_k}^{1/d}} \right), \quad s \in [0, 1)^d.$$

We shall first show that, for any choice $\delta > 0$, we have almost surely

$$\limsup_{n \rightarrow \infty} \sup_{1 \leq i \leq m_{n_k}} \inf_{g \in \Gamma_{cf(z_l)}} \|\mathcal{H}_n(z_{i, n_k}, \cdot) - g\| \leq 2\epsilon. \quad (5.11)$$

Consider the following probabilities for all large k .

$$\mathbb{P}_k := \mathbb{P} \left(\bigcup_{1 \leq i \leq m_{n_k}} \bigcup_{n \in N_k} \mathcal{H}_n(z_{i, n_k}, \cdot) \notin \Gamma_{cf(z_l)}^{2\epsilon} \right).$$

We have, ultimately as $k \rightarrow \infty$,

$$\mathbb{P}_k \leq m_k \max_{1 \leq i \leq m_{n_k}} \mathbb{P} \left(\bigcup_{n \in N_k} \mathcal{H}_n(z_{i, n_k}, \cdot) \notin \Gamma_{cf(z_l)}^\epsilon \right). \quad (5.12)$$

We now make use of a well-known maximal inequality (see, e.g., Deheuvels and Mason [5], Lemma 3.4) to get, for all large k and for all $1 \leq i \leq m_{n_k}$,

$$\mathbb{P} \left(\bigcup_{n \in N_k} \mathcal{H}_n(z_{i, n_k}, \cdot) \notin \Gamma_{cf(z_l)}^{2\epsilon} \right) \leq 2\mathbb{P} \left(\mathcal{H}_{n_k}(z_{i, n_k}, \cdot) \notin \Gamma_{cf(z_l)}^\epsilon \right). \quad (5.13)$$

We point out that the conditions of Lemma 3.4 in [5] are satisfied since, by a straightforward use of Markov's inequality we have, ultimately as $k \rightarrow \infty$,

$$\sup_{z \in H} \max_{n \in N_k} \mathbb{P}(\|\mathcal{H}_{n_k}(z, \cdot) - \mathcal{H}_n(z, \cdot)\| \geq \epsilon) \leq \frac{1}{2}.$$

Making use of (5.13) in (5.12), we obtain, for all large k ,

$$\begin{aligned} \mathbb{P}_k &\leq 2m_k \max_{1 \leq i \leq m_{n_k}} \mathbb{P} \left(\mathcal{H}_{n_k}(z_{i, n_k}, \cdot) \notin \Gamma_{cf(z_l)}^\epsilon \right) \\ &= 2m_{n_k} \max_{1 \leq i \leq m_{n_k}} \mathbb{P} \left(\Delta_{n_k}(z_{i, n_k}, h_{n_k}, \cdot) \notin \Gamma_{cf(z_l)}^\epsilon \right) \\ &\leq 4m_{n_k} \max_{1 \leq i \leq m_{n_k}} \mathbb{P} \left(\Delta \Pi_{n_k}(z_{i, n_k}, h_{n_k}, \cdot) \notin \Gamma_{cf(z_l)}^\epsilon \right). \end{aligned} \quad (5.14)$$

The last inequality is a consequence of usual poissonization techniques (see, e.g., Mason [10], Fact 6). We now make use of Proposition 3.1, which, together with (5.2) leads to the following inequality, ultimately as $k \rightarrow \infty$,

$$\mathbb{P}_k \leq 4m_{n_k} \max_{1 \leq i \leq m_k} \exp \left(-\frac{f(z_{i, n_k})}{f(z_l)} (1 + 2\alpha_{z_l}) \log n_k \right).$$

Moreover (5.4) entails $\mathbb{P}_k \leq 4m_{n_k} \exp(-(1 + \alpha_{z_l}) \log n_k)$. Since $m_{n_k} = h_{n_k}^{-1+o(1)} = n_k^{1+o(1)}$ as $k \rightarrow \infty$ (recall (5.10)), the sumability of \mathbb{P}_k follows, which proves (5.11) by the Borel-Cantelli lemma. We point out that (5.11) is true whatever the choice of $\delta > 0$ (recall (5.9)). We now focus on showing that, for a small value of $\delta > 0$ we have

$$\limsup_{k \rightarrow \infty} \sup_{z \in H_{z_l}} \min_{1 \leq i \leq m_{n_k}} \max_{n \in N_k} \|\mathcal{H}_n(z_{i,n_k}, \cdot) - \Delta_n(z, h_n, \cdot)\| \leq 7\epsilon \text{ a.s.}, \quad (5.15)$$

which will be achieved through two separate lemmas.

Lemma 5.2 *Assume that the conditions of Theorem 2 are fulfilled. There exists $\delta_\epsilon > 0$ such that, for any choice of $0 < \delta < \delta_\epsilon$ we have almost surely*

$$\limsup_{k \rightarrow \infty} \max_{n \in N_k} \max_{1 \leq i \leq m_{n_k}} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| \leq \epsilon.$$

Proof: For all large k we have

$$\begin{aligned} & \mathbb{P} \left(\max_{n \in N_k} \max_{1 \leq i \leq m_{n_k}} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| > \epsilon \right) \\ &= \mathbb{P} \left(\bigcup_{1 \leq i \leq m_{n_k}} \bigcup_{n \in N_k} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| > \epsilon \right) \\ &\leq m_{n_k} \max_{i \leq m_{n_k}} \mathbb{P} \left(\bigcup_{n \in N_k} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| > \epsilon \right) \end{aligned} \quad (5.16)$$

Fix $k \geq 1$, $1 \leq i \leq m_{n_k}$ and $z \in z_{i,n_k} + (\delta h_{n_k})^{1/d} [0, 1)^d$. We write $z_{i,n_k} := (z_{i,n_k}^1, \dots, z_{i,n_k}^d)$, $z := (z^1, \dots, z^d)$ and $Z_j := (Z_j^1, \dots, Z_j^d)$, $j \geq 1$. Notice that for each $p = 1, \dots, d$ we have $z_{i,n_k}^p \leq z^p \leq z_{i,n_k}^p + (\delta h_{n_k})^{1/d}$. Hence, in virtue of the equality $|1_A - 1_B| = 1_{A-B} + 1_{B-A}$ we have, for each integer j we have almost surely, for each $(s_1, \dots, s_d) \in [0, 1)^d$,

$$\begin{aligned} & \left| 1_{[0,s)} \left(\frac{Z_j - z}{h_{n_k}^{1/d}} \right) - 1_{[0,s)} \left(\frac{Z_j - z_{i,n_k}}{h_{n_k}^{1/d}} \right) \right| \\ &= 1_{\left\{ [z, z + h_{n_k}^{1/d} s) - [z_{i,n_k}, z_{i,n_k} + h_{n_k}^{1/d} s) \right\}}(Z_j) + 1_{\left\{ [z_{i,n_k}, z_{i,n_k} + h_{n_k}^{1/d} s) - [z, z + h_{n_k}^{1/d} s) \right\}}(Z_j) \\ &\leq \sum_{l=1}^d 1_{\left[z_{i,n_k}^l + s_l h_{n_k}^{1/d}, z_{i,n_k}^l + h_{n_k}^{1/d} (s_l + \delta^{1/d}) \right]}(Z_j^l) \prod_{1 \leq p \neq l \leq d} 1_{\left[z_{i,n_k}^p, z_{i,n_k}^p + h_{n_k}^{1/d} (s_p + \delta^{1/d}) \right]}(Z_j^p) \\ &\quad + \sum_{l=1}^d 1_{\left[z_{i,n_k}^l, z_{i,n_k}^l + (\delta h_{n_k})^{1/d} \right]}(Z_j^l) \prod_{1 \leq p \neq l \leq d} 1_{\left[z_{i,n_k}^p, z_{i,n_k}^p + h_{n_k}^{1/d} s_p \right]}(Z_j^p) \end{aligned} \quad (5.17)$$

$$=: X_{j,k,i,\delta}(s). \quad (5.18)$$

Here (5.17) follows from $z_{i,n_k}^l \leq z^l \leq z_{i,n_k}^l + \delta^{1/d} h_{n_k}^{1/d}$, $l = 1, \dots, d$. As the $X_{j,k,i,\delta}(\cdot)$ are positive processes almost surely, (5.18) entails, for all large k and for all $1 \leq i \leq m_{n_k}$,

$$\begin{aligned}
& \mathbb{P}\left(\bigcup_{n \in N_k} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| > \epsilon\right) \\
& \leq \mathbb{P}\left(\bigcup_{n \in N_k} \sup_{z \in C_{i,n_k}} \sup_{s \in [0,1]^d} \left| \sum_{j=1}^n 1_{[0,s)} \left(\frac{Z_j - z}{h_{n_k}^{1/d}} \right) - 1_{[0,s)} \left(\frac{Z_j - z_{i,n_k}}{h_{n_k}^{1/d}} \right) \right| \geq \epsilon c f(z_{i,n_k}) \log n_k\right) \\
& \leq \mathbb{P}\left(\bigcup_{n=1}^{n_k} \sup_{s \in [0,1]^d} \sum_{j=1}^n X_{j,k,i,\delta}(s) \geq \epsilon c f(z_{i,n_k}) \log n_k\right) \\
& \leq \mathbb{P}\left(\left\| \sum_{j=1}^{n_k} X_{j,k,i,\delta}(\cdot) \right\| \geq \epsilon c f(z_{i,n_k}) \log n_k\right). \tag{5.19}
\end{aligned}$$

But a close look at (5.17) leads to the conclusion that, almost surely, for each $s \in [0,1]^d$,

$$\begin{aligned}
0 & \leq \sum_{j=1}^{n_k} X_{j,k,i,\delta}(s) \\
& \leq 2dc f(z_{i,n_k}) \log n_k \sup_{\substack{s, s' \in [0,2]^d, \\ \|s' - s\|^d < \delta}} \left| \Delta_{n_k}(z_{i,n_k}, h_{n_k}, s') - \Delta_{n_k}(z_{i,n_k}, h_{n_k}, s) \right| \tag{5.20}
\end{aligned}$$

Here we have written $\|s\|_d := \max\{|s_j|, j = 1, \dots, p\}$. Now (5.20) together with (5.19) entails

$$\begin{aligned}
& \frac{1}{2} \mathbb{P}\left(\bigcup_{n \in N_k} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| > \epsilon\right) \\
& \leq \frac{1}{2} \mathbb{P}\left(\sup_{\substack{s, s' \in [0,2]^d, \\ \|s' - s\|_d < \delta^{1/d}}} \left| \Delta_{n_k}(z_{i,n_k}, h_{n_k}, s') - \Delta_{n_k}(z_{i,n_k}, h_{n_k}, s) \right| > \frac{\epsilon}{2d}\right) \\
& \leq \mathbb{P}\left(\sup_{\substack{s, s' \in [0,1]^d, \\ \|s' - s\|_d < \frac{\delta^{1/d}}{2}}} \left| \Delta \Pi_{n_k}(z_{i,n_k}, h_{n_k}, 2s') - \Delta \Pi_{n_k}(z_{i,n_k}, h_{n_k}, 2s) \right| > \frac{\epsilon}{2d}\right) \tag{5.21}
\end{aligned}$$

Here (5.21) follows from poissonization techniques. Now consider the following sequence $\mathfrak{h}_n := 2^d h_n$, $n \geq 1$. Clearly, $(\mathfrak{h}_n)_{n \geq 1}$ satisfies (HVE1) and (HVE2),

replacing c by $\mathfrak{c} := 2^d c$. Moreover, for each $k \geq 1$, $1 \leq i \leq m_{n_k}$ we have almost surely, for all $s \in [0, 1)^d$,

$$\Delta \Pi_{n_k}(z_{i,n_k}, h_{n_k}, 2s) = \Delta \Pi_{n_k}(z_{i,n_k}, \mathfrak{h}_{n_k}, s). \quad (5.22)$$

Applying Proposition 3.1 we deduce that the triangular array of processes

$$U_{k,i}(\cdot) := \Delta \Pi_{n_k}(z_{i,n_k}, \mathfrak{h}_{n_k}, 2\cdot), \quad k \geq 1, \quad 1 \leq i \leq m_{n_k}$$

satisfies the ULDP in $(B([0, 1)^d), \|\cdot\|)$ (see §2) for the rate function I and for the following triangular array:

$$\epsilon_{k,i} := (c2^d f(z_{i,n_k}) \log n_k)^{-1} k \geq 1, \quad 1 \leq i \leq m_{n_k}.$$

Now consider the following set

$$\Gamma := \left\{ g \in \mathcal{M}([0, 1)^d), \quad I(g) \leq \frac{4}{2^d c \beta} \right\}.$$

By proposition 2.1, there exists $\delta_\epsilon > 0$ such that

$$\sup_{g \in 2^d \Gamma} \sup_{s, s' \in [0, 2)^d, \|s' - s\|_d \leq \delta_\epsilon^d / 2} |g(s') - g(s)| < (4d)^{-1} \epsilon. \quad (5.23)$$

Now choose $0 < \delta < \delta_\epsilon$ arbitrarily for the construction of the z_{i,n_k} , $k \geq 1, 1 \leq i \leq m_{n_k}$ (recall (5.9)). By lower-semicontinuity of I , the closed set

$$F := \left\{ g \in \mathcal{M}([0, 2)^d), \quad \inf_{g' \in \Gamma} \|g - g'\|_{[0, 2)^d} \geq \frac{2^{-d} \epsilon}{8d} \right\}$$

satisfies $I(F) > 4/(2^d c \beta)$. Hence, (5.21) together with (5.23) leads to the following inequalities for all large k and for each $1 \leq i \leq m_{n_k}$.

$$\begin{aligned} & \mathbb{P} \left(\bigcup_{n \in N_k} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| > \epsilon \right) \\ & \leq 2\mathbb{P} \left(\sup_{\substack{s, s' \in [0, 1)^d, \\ \|s' - s\|_d < \delta^{1/d}/2}} |U_{k,i}(s') - U_{k,i}(s)| > \epsilon(2d)^{-1} \right) \\ & \leq 2\mathbb{P}(\Delta \Pi_{n_k}(z_{i,n_k}, \mathfrak{h}_{n_k}, \cdot) \in F) \\ & \leq 2 \exp \left(-\frac{3}{4} I(F) \mathfrak{c} f(z_{i,n_k}) \log n_k \right) \\ & \leq 2 \exp \left(-3 \times \frac{c2^d f(z_{i,n_k})}{\beta c 2^d} \log n_k \right) \\ & \leq 2 \exp(-3 \log n_k). \end{aligned} \quad (5.24)$$

Now (5.24) in combination with (5.16) entails, for all large k ,

$$\mathbb{P} \left(\max_{n \in N_k} \max_{1 \leq i \leq m_{n_k}} \sup_{z \in C_{i,n_k}} \left\| \mathcal{H}_n(z_{i,n_k}, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| > \epsilon \right) \leq \frac{2m_{n_k}}{n_k^3} \quad (5.25)$$

But for fixed $\delta > 0$ we have $m_{n_k} = h_{n_k}^{-1+o(1)} = n_k^{1+o(1)}$ as $k \rightarrow \infty$. The proof of Lemma 5.2 is concluded by applying the Borel-Cantelli lemma to (5.25). \square

Lemma 5.3 *Under the assumptions of Theorem 2, for any choice of $\delta > 0$, we have almost surely*

$$\limsup_{k \rightarrow \infty} \max_{1 \leq i \leq m_{n_k}} \sup_{z \in C_{i,n_k}} \max_{n \in N_k} \left\| \Delta_n(z, h_n, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| \leq 6\epsilon.$$

Proof: For all large k and for all $1 \leq i \leq m_{n_k}$, $z \in C_{i,n_k}$, $n \in N_k$ we have almost surely, for each $s \in [0, 1]^d$,

$$\Delta_n(z, h_n, s) = T_{n,i,k} \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \rho_{n,k}s), \quad (5.26)$$

with $T_{i,n,k} := f(z_{i,n_k}) \log n_k / f(z) \log n$ and $\rho_{n,k}^d := h_{n_k} / h_n$. First notice that

$$\lim_{k \rightarrow \infty} \max_{1 \leq i \leq m_{n_k}} \sup_{z \in C_{i,n_k}} |T_{n,i,k} - 1| = 0, \quad \lim_{k \rightarrow \infty} \max_{n \in N_k} |\rho_{n,k} - 1| = 0.$$

Moreover, by Proposition 2.1 we have

$$\lim_{T \rightarrow 1, \rho \rightarrow 1} \sup_{g \in \Gamma_{cf(z_l)}} \|Tg(\rho^{1/d} \cdot) - g(\cdot)\| = 0. \quad (5.27)$$

Finally, by (5.11) and by Lemma 5.2 we have, for all large k and for all $1 \leq i \leq m_{n_k}$, $z \in C_{i,n_k}$, $n \in N_k$,

$$\inf_{g \in \Gamma_{cf(z_l)}} \left\| \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) - g \right\| < 3\epsilon \text{ almost surely.} \quad (5.28)$$

Hence, combining (5.26), (5.27), (5.27), (5.28) and the triangle inequality, we obtain almost surely, for all large k and for all $n \in N_k$:

$$\begin{aligned} & \left\| \Delta_n(z, h_n, \cdot) - \frac{f(z)}{f(z_{i,n_k})} \mathcal{H}_n(z, \cdot) \right\| \\ & \leq 6\epsilon, \end{aligned}$$

which proves Lemma 5.3. \square

End of the proof of part(ii) of Theorem 2: By combining Lemma 5.3 with Lemma 5.2 we conclude that (5.15) is true for $\delta > 0$ small enough. Now (5.15) together with (5.11) leads to

$$\limsup_{n \rightarrow \infty} \sup_{z \in H_{z_l}} \inf_{g \in \Gamma_{cf(z_l)}} \|\Delta_n(z, h_n, \cdot) - g\| \leq 9\epsilon \text{ almost surely.}$$

Whence, recalling (5.5),

$$\limsup_{n \rightarrow \infty} \sup_{z \in H_{z_l}} \inf_{g \in \Gamma_{cf}(z)} \|\Delta_n(z, h_n, \cdot) - g\| \leq 10\epsilon \text{ almost surely.} \quad (5.29)$$

Repeating (5.29) for each $l = 1, \dots, L$ (recall (5.7)) we get

$$\limsup_{n \rightarrow \infty} \sup_{z \in H} \inf_{g \in \Gamma_{cf}(z)} \|\Delta_n(z, h_n, \cdot) - g\| \leq 10\epsilon \text{ almost surely.}$$

As $\epsilon > 0$ was chosen arbitrarily, the proof of part(ii) of Theorem 2 is concluded. \square

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